

Problem: The relationship between diffusion-weighted images (DWIs) and the (apparent) diffusion tensor (DTI) is

$$(1) \quad I^{(q)} = J \exp(-\mathbf{b}^{(q)} \bullet \mathbf{D}) + \text{noise}^{(q)}$$

where $I^{(q)}$ = image intensity for the q^{th} DWI ($q=0,1,\dots$), $\mathbf{b}^{(q)}$ = weighting matrix [e.g., $b_{ij}^{(q)} = \gamma^2 G_i^{(q)} G_j^{(q)} \delta^2 (\Delta - \delta / 3)$],

\mathbf{D} = diffusion tensor (unknown), J = "true" image intensity for $\mathbf{b}=\mathbf{0}$ (unknown), and $\mathbf{b} \bullet \mathbf{D} \equiv \sum_{i,j} b_{ij} D_{ij}$. The simplest way to estimate the diffusion tensor from diffusion-weighted images is to take the logarithm of (1) and ignore the noise:

$$(2) \quad -\log(I^{(q)} / J) = \mathbf{b}^{(q)} \bullet \mathbf{D}$$

which is a set of linear relationships between the DWI data ($I^{(q)}$), scan parameters ($\mathbf{b}^{(q)}$), and the diffusion tensor (\mathbf{D}) — assuming that J is known (usually $J = I^{(0)}$, acquired with $\mathbf{b}^{(0)} = \mathbf{0}$); \mathbf{D} is solved for in (2) by linear least squares.

One difficulty with the log-linear approach is that the resulting \mathbf{D} might not be positive definite (p.d.) [1]. A second difficulty is that the noise is not being treated properly [2] — linear least squares is appropriate when the noise is additive and each sample has the same variance; taking the logarithm violates both suppositions. When the eigenvalues of \mathbf{D} are significantly disparate, these violations often result in poor estimates for \mathbf{D} . We instead choose to fit \mathbf{D} directly to (1), using a method sure to return a p.d. matrix. Our method extends Tschumperle [3] to deal with (1) rather than (2).

Solution: The goal is to find the value J and the symmetric p.d. matrix \mathbf{D} that minimize the weighted error functional

$$(3) \quad E(\mathbf{D}, J) = \frac{1}{2} \sum_q w_q \left[J \exp(-\mathbf{b} \bullet \mathbf{D}) - I^{(q)} \right]^2$$

As J appears quadratically, it can be estimated directly: $\hat{J}(\mathbf{D}) = \left[\sum_q w_q I^{(q)} \exp(-\mathbf{b}^{(q)} \bullet \mathbf{D}) \right] / \left[\sum_q w_q \exp(-2\mathbf{b}^{(q)} \bullet \mathbf{D}) \right]$. We use

a modified gradient descent method to compute $\hat{\mathbf{D}}$ (minimizer of E). The gradient matrix of E wrt \mathbf{D} is

$$\mathbf{F} = -\sum_q w_q \left[J \exp(-\mathbf{b}^{(q)} \bullet \mathbf{D}) - I^{(q)} \right] \mathbf{b}^{(q)}.$$

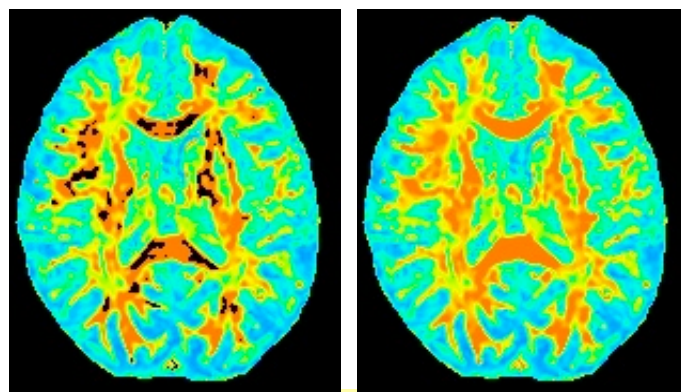
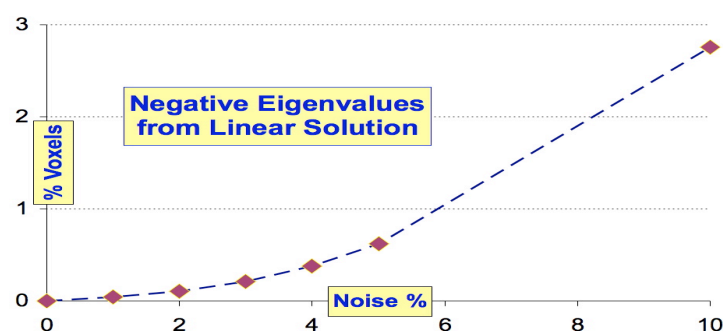
Pure gradient descent would solve the differential equation $\partial \mathbf{D}(s) / \partial s = -\mathbf{F}(\mathbf{D})$,

initialized with $\mathbf{D}(s=0)$ calculated via (2). However, this method of minimizing (3) often leads to an indefinite \mathbf{D} . We use instead the fastest descent direction linear in \mathbf{F} that *guarantees* \mathbf{D} remains p.d., by solving

$\partial \mathbf{D}(s) / \partial s = -[\mathbf{F}(\mathbf{D})\mathbf{D}^2 + \mathbf{D}^2\mathbf{F}(\mathbf{D})]$; along this curve, $\partial E / \partial s = -2\|\mathbf{F}\mathbf{D}\|^2$. The descent curve is computed using a Padé approximant method consistent with this equation, which also ensures \mathbf{D} remains p.d. even for finite stepsizes: define

$\mathbf{H}_{\pm}(\epsilon) = \mathbf{I} \pm \frac{1}{2} \epsilon \mathbf{F} \mathbf{D}$, and then $\mathbf{D}(s + \epsilon) = \mathbf{H}_{-}(\epsilon) \mathbf{H}_{+}(\epsilon)^{-1} \mathbf{D}(s) \mathbf{H}_{+}(\epsilon)^{-1} \mathbf{H}_{-}(\epsilon)$. The stepsize ϵ is chosen as large as possible, but ensuring that $E(s)$ is decreasing. After convergence, we calculate the residuals; the initial weights are modified to down-weight outlier $I^{(q)}$ values, and the descent is restarted. The result is an efficient nonlinear robust [4] positive definite \mathbf{D} estimator. Free C software is included in AFNI.

Results (Is a picture is worth 1000 words?)



Fractional anisotropy with linear (LEFT) and nonlinear (RIGHT) methods: voxels with negative eigenvalues are black.

References:

- [1] Skare et al, *Mag.Reson.Imaging*, **18**:659-669 (2000).
- [2] Jones & Basser, *Magn.Reson.Med.*, **52**:979-993 (2004).
- [3] Tschumperle & Deriche, *Lect.Notes Comp.Sci.* **2809**:530-541 (2003).
- [4] Mangin et al., *Med.Imag.Analysis* **6**:191-198 (2003)