The Geometry Factor as a Cramér-Rao Bound for Magnitude and Phase

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Introduction: In SENSE imaging, aliased data from multiple coils is used to estimate the unaliased components of the image. Noise propagation from the aliased coil images into the complex reconstructed images is typically quantified by the geometry factor (g-factor) [1]. The g-factor is the ratio of the noise in the complex reconstructed image of the accelerated acquisition over the noise in the full acquisition when using an algorithm that minimizes the squared error [1,2]. The Cramér-Rao bound (CRB) can be computed to provide the minimum variance of the magnitude and phase[3]. Phase estimation is important for applications like flow quantitation but the noise in the phase in parallel imaging has not been characterized in the same way as the magnitude, in part because the phase is a nonlinear parameter of the measurements. The objective of this abstract is to analyze the noise amplification for the phase and to make an explicit connection between the g-factor and the CRB for magnitude and the phase.

Theory and Methods: We recast SENSE in terms of the magnitude and phase [3,4] assuming that the sensitivity matrix has no errors: $\mathbf{g} = \mathbf{A}(\boldsymbol{\varphi})\boldsymbol{\rho} + \boldsymbol{\varepsilon}$ (Eqn 1), where $\mathbf{g}$ is the aliased coil data, $\mathbf{A}$ is the system matrix, $\boldsymbol{\varphi}$ is the unaliased image phase vector , $\boldsymbol{\rho}$ is the unaliased image magnitude vector and $\boldsymbol{\varepsilon}$ is the complex Gaussian noise in the measurements (uncorrelated and with equal variance σ_ε^2). We consider a 4 coil acquisition with a 2x acceleration factor. In which case, the equation above for a particular pixel becomes:

$$\mathbf{A}(\boldsymbol{\varphi}) = \begin{pmatrix} |c_{11}|\cos(\omega_{11}+\varphi_1) & |c_{12}|\cos(\omega_{12}+\varphi_2) \\ |c_{11}|\sin(\omega_{11}+\varphi_1) & |c_{12}|\sin(\omega_{12}+\varphi_2) \\ \vdots & \vdots \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} Re\{I_1\} \\ Im\{I_1\} \\ \vdots \end{pmatrix} \text{ coil images}, \quad \boldsymbol{\rho} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \text{ magnitudes}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} Re\{\varepsilon_1\} \\ Im\{\varepsilon_1\} \\ \vdots \end{pmatrix} \text{complex noise} \qquad \text{(Eqn 2)}$$

where c_{nm} are the elements of the coil sensitivity matrix (C) with magnitude $|c_{nm}|$ and phase ω_{nm}. Given any equation of the form given by Eqn. 1, reference [3] gives an expression for the Fisher Information Matrix (F_{kl}) which can be inverted to obtain the CRB. To shorten the notation, we place our unknowns in a vector $\boldsymbol{\theta} = [\rho_1, \rho_2, \varphi_1, \varphi_2]$ and use the symmetry of F, with k and l as the row and column indexes:

$$F(k=1,2,l=1,2) = \frac{1}{\sigma_\varepsilon^2}\left[A^t A\right], \; F(k=1,2,l=3,4) = \frac{1}{\sigma_\varepsilon^2}\left[A\frac{\partial A}{\partial \theta_l}\boldsymbol{\rho}\right]_k, \; F(k=3,4,l=3,4) = \frac{1}{\sigma_\varepsilon^2}\boldsymbol{\rho}^t\frac{\partial A^t}{\partial \theta_l}\frac{\partial A}{\partial \theta_k}\boldsymbol{\rho}.$$

The variance of the unknowns (θ_k) is bounded by the CRB: $\sigma_{\theta_k}^2 \geq \left[F^{-1}\right]_{kk}$. In an analogous way to the g-factor, we normalize the CRB of the unaliasing problem by the CRB of the case where we have all four coil measurements but the pixel has no aliasing (*CRBN*), which we can express directly from the coil sensitivity matrix:

$CRBN(\rho_k) = \dfrac{1}{\left[C^H C\right]_{kk}}$, $CRBN(\varphi_k) = \dfrac{1}{\rho_k^2\left[C^H C\right]_{kk}}$, as we would expect the variance of the phase depends on the length of the associated magnitude vector. Using this normalization, we define a noise efficiency: $E(\theta_k) = \dfrac{CRB(\theta_k)}{CRBN(\theta_k)}$, that quantifies the uncertainty associated with the reconstruction process for all unknowns. For completeness we include the expression for the g-factor for the kth pixel: $g_k = \sqrt{\left[\left(C^H C\right)^{-1}\right]_{kk}\left(C^H C\right)_{kk}}$, the g-factor is phrased in terms of the standard deviation, not the variance, hence in our comparisons, we will use the square root of the noise efficiency.

Results: We used a sensitivity map from a 4 coil acquisition for a 2x acceleration acquired on a GE Signa 1.5T scanner to compute both the g-factor and the normalized CRB for the magnitude and phase. Fig. 1 shows the aliasing which is being removed by the reconstruction process (Eqn. 1). Fig. 2 shows the g-factor map, and the square root of the noise efficiency for magnitude and phase. (with a mask of ones in the areas where there is no aliasing). In our numerical experiments, these three quantities were the same.

Discussion: From its initial formulation, the g-factor has been a useful measure of noise amplification. The explicit extension to the noise amplification of the phase images provides an interpretation that ensures that minimizing the g-factor optimizes phase estimation. The CRB provides a pixel-wise bound on the mean square error and can be used to evaluate the error in reconstruction algorithms [2,3]. The numerical results shown suggest an alternative interpretation of the g-factor as a normalized CRB for both magnitude and phase. An analytic solution and extension to sensitivity maps with errors are the focus of our future work.

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References: 1. Pruessmann et al, MRM, 1999, 42: 952-962, 2. Scheffe, et. al. ISBI 2004, 213-216,. 3. Pineda et al, MRM, 2005, 54, 625-625., 4. Lew et al, ISRMR 2005, pg.2443.

Figure 1. aliased magnitude image from a single coil

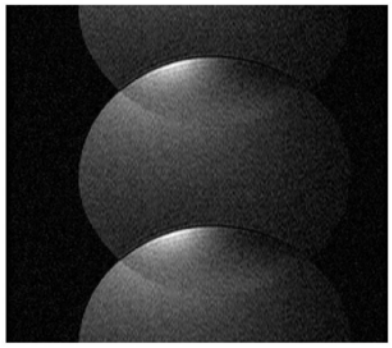
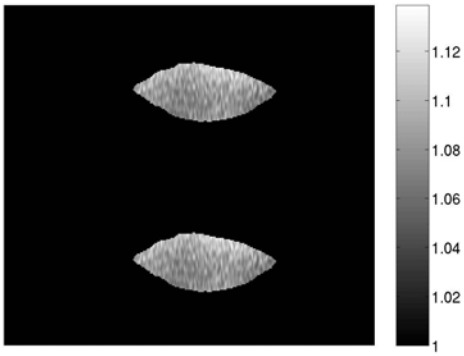
Figure 2. g-factor, square root of normalized CRB for magnitude and phase