An iterative method for fast regularized parallel MRI reconstruction

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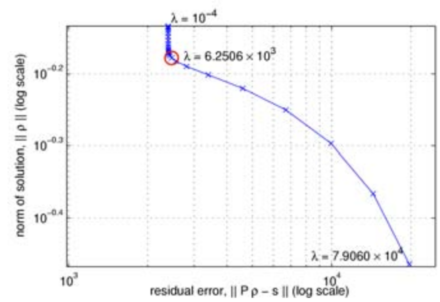
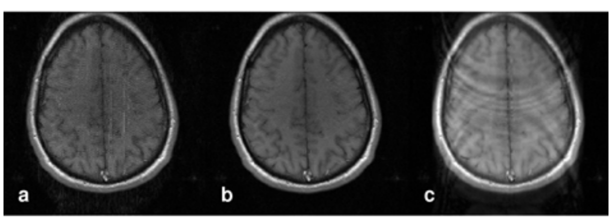
Introduction: Reconstruction of non-uniformly subsampled parallel MRI (pMRI) data typically imposes significantly higher computational costs over more commonly used pMRI methods, such as SENSE or GRAPPA. This is due primarily to the larger linear systems of equations one needs to solve in order to reconstruct the image, and the need for system regularization due to poor system conditioning. Because direct inversion of the linear system is often impractical, iterative approaches, such as the method of conjugate gradients (CG) are often employed [1].

Recently, Kilmer and O'Leary proposed an efficient method to rapidly find both a good regularization parameter value and multiple solutions for iterative least squares solvers of damped-least-squares (DLS) problems [2]. This method exploits the fact that the Krylov subspace formed by iterative solvers is in fact independent of the regularization parameter for regularization with a scaled identity matrix. We present here a modified version of the LSQR-Hybrid algorithm described in [2] to handle reconstruction of non-uniformly sampled, complex-valued parallel MRI data. This implementation utilizes the fast matrix-vector products enabled by the particular structure of the parallel MRI reconstruction problem, as in [1]. We achieve a reduction in reconstruction time over naive implementations by around two orders of magnitude: one from the LSQR-Hybrid L-curve generation, and the other from the fast matrix-vector products.

Theory: The LSQR algorithm solves $\min_\rho |P\rho\text{-}s|_2^2$ directly while the CG algorithm instead solves the associated system of normal equations, $P^HPx=P^Hs$. If P is full rank, there is a unique solution to the least squares problem, which is also the solution to the normal equations, and vice versa. The inclusion of damped-least-squares regularization adds a second term to the minimization problem: $\min_\rho\{|P\rho\text{-}s|_2^2+\lambda^2|\rho|_2^2\}$ where the scalar λ is referred to as the regularization parameter. With non-zero λ, a unique solution exists. The selection of a good value for λ is non-trivial. To use the L-curve technique [3], one constructs multiple solutions of the reconstruction problem across a range of regularization parameter values, and plots the residual error, $|P\rho\text{-}s|$, against the solution norm, $|\rho|$, in a log-log plot. This function is parametric in λ and typically results in an L-shaped curve. The inflection point on this curve corresponds to a good tradeoff between the residual error and noise in the solution, and guides the regularization parameter choice. Both the CG and LSQR algorithms operate over the same underlying Krylov subspace, $K_k=\text{span}\{P^Hs,(P^HP)\,P^Hs,\ldots,(P^HP)^{k\text{-}1}P^Hs\}$, and the solution ρ_k at iteration k lies in this space. The key to efficient L-curve generation with the LSQR-Hybrid algorithm is the fact that the Krylov subspace is not affected by the value of the regularization parameter λ, since the Krylov subspace does not change with the addition of a scalar multiple of I to P^HP. In the LSQR-Hybrid algorithm [2], one implicitly solves a regularized projected problem. One can compute the quantities needed for the L-curve for this smaller problem without explicitly computing the solution to that system. It turns out that this L-curve is identical to the L-curve for the original system. This opportunity to embed an L-curve iteration within each iteration of LSQR leads to significant computational savings over calculating an L-curve using a non-hybrid approach.

Results: We tested in Matlab two algorithms to solve the SPACE RIP formulation [4]: a CG algorithm with fast matrix-vector multiplications but without embedded L-curve computation, and an LSQR algorithm with both improvements, on data acquired using a Fast Spin Echo (TR = 500 ms, TE = 13.8 ms) sequence on a 1.5T GE Signa EXCITE 11.0 MR scanner with an 8-reciever phased-array head coil. The non-uniform sub-sampling pattern used followed an exponentially weighted distribution, $Z_{ed}(y)=\exp\{-0.87|y|\}$, [5], with an overall acceleration (down-sampling) factor of 3x. The regularization parameter λ was automatically chosen for each column using the embedded L-curve approach for all reconstructions. For the LSQR-Hybrid algorithm, all the required values were obtained with one pass through the iterative solution, while for CG SPACE RIP one set of iterations was computed for each value of λ. The L-curve (right) shows the L-curve for one column of the reconstruction. The inflection point of the L-curve is identified by the circle in the figure and corresponds to the highest value of regularization that suppressed noise in the reconstruction without introducing visibly objectionable smoothing and/or aliasing artifacts.





$\lambda = 1\times10^{-4}$ $\lambda = 6.2506\times10^{4}$ $\lambda = 7.9060\times10^{4}$

The figure at left shows images corresponding to three values of λ. In panel (a), the solution is under-regularized and we see significant noise amplification. Panel (b) shows the reconstruction for the value of λ at the inflection point of the L-curve. Noise is no longer present, while aliasing artifacts are minimal. At higher levels of regularization, panel (c), the regularization is excessive. Visually important modes from acquisition encoding are suppressed, resulting in smoothed anatomical detail and increased aliasing artifacts.

The table shows the relative reconstruction times between the LSQR-Hybrid and CG algorithm. These calculations were performed on a moderate performance 1GHz PIII with 512MB of RAM. They clearly illustrate the computational savings provided by the LSQR-Hybrid approach when solutions with multiple λ's are found.

Number of λ solutions:	1	5	10	15	25	40	50
LSQR recon time (sec):	14	16	20	22	30	37	45
CG recon time (sec):	9	49	97	144	240	417	518

Conclusion: In this paper we compared two iterative algorithms, CG and LSQR, to reconstruct non-uniform subsampled pMRI data using SPACE RIP. The linear system matrix associated with non-uniform subsampling typically has a high condition number and the reconstruction algorithm can benefit from regularization. Through use of the LSQR-Hybrid algorithm, we can achieve significant computational savings when the regularization parameter value is automatically selected using the L-curve. This hybrid L-curve approach provides robust regularization with minimal additional computation time.

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