# Voronoi based sampling density correction

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## Introduction

By its very nature, image reconstruction from non-uniformly sampled Fourier Transform data requires compensation of the sampling density for accurate results. In the usual gridding algorithm, [1], the non-uniform sampling is accounted for by data pre-multiplication. In least square based methods, the non-uniformity is corrected through a weighting factor [2]. Among the several methods proposed in the literature for sampling density compensation, the approach based on Voronoi diagram is the most natural. In this approach, the continuous transformation from K-space to the image space (R-space) is approximated by a discrete expression, $I(\vec{r}) = \int d^n\vec{\kappa}\, S(\vec{\kappa}) e^{2\pi i \vec{\kappa}\vec{r}} \approx \sum w(\vec{\kappa}) S(\vec{\kappa}) e^{2\pi i \vec{\kappa}\vec{r}}$. The factor $w(\vec{\kappa})$ is the measure of the domain in K-space closer to the given point than to any other sampling location. The geometric definition makes the calculation amenable to a general-purpose implementation, separating the data content from the geometry of the K-space trajectory. However, the complexity of the Voronoi decomposition increases with the dimensionality, making the use of this technique difficult to implement efficiently in 3D. In this work, we present a 3D implementation of the Voronoi-based density compensation, using the Computational Geometry Algorithms Library (CGAL) library.

## Theory

The underlying data structure implemented by CGAL, namely the Delaunay hierarchy of triangulations (in short a triangulation) allows for fast sequential insertion of the K-space points. This sophisticated data structure, allows the simultaneous clustering of the data points. In our approach this step is performed by augmenting the Delaunay structure with the measured value at that location, as well as with a count of points composing the cluster. By choosing a small distance $\delta$ (any two points closer than this, are considered as part of a repeated measurement) a N-step algorithm is performed. At each step a new data point (signal s, K-space position p) is either added as a new point in the triangulation or the existent location 'i', characterized by $|\vec{\kappa}_i - \vec{p}| < \delta$ is updated according to $\vec{\kappa}_i \leftarrow (n_i\vec{\kappa} + \vec{p})/(n_i+1) \quad n_i \leftarrow n_i + 1 \quad s_i \leftarrow (n_i s_i + s)/(n_i+1)$.

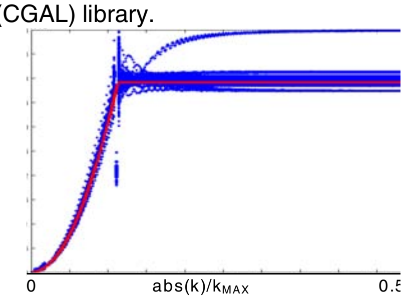


***Figure 1**: Voronoi volume vs. the radial distance, numerically computed (dots) and theoretically predicted (solid line).The gradient discretization and the finite sampling rates may introduces discrepancies between the ideal and the real values.*

At the end of this procedure, various geometric queries can be performed (due to the underlying CGAL implementation), the computation of the Voronoi volume being one of them.

A meaningful computation for image reconstruction requires finite weights, only, which involves the transformation of the true unbounded Voronoi decomposition into a bounded one. The enclosure techniques, (i.e. those using extra points, to separate the measurement points from the "point at infinity") do perturb the finite weights, too. We advocate an approximate solution. We advocate an approximate solution. In terms of a regularization parameter $\alpha$<1., and a chosen geometrical criteria (e.g. distance to the origin less than a prescribed bound) the points are partitioned as "good" or "bad" points, such that the good points fill a fraction $\alpha V_{TOT}$ of the K-space domain. For those points, the weight is assigned to the Voronoi volume. The $N_{rest}$ "bad" points receive equal shares of the reminder, their weights being assigned to $(1-\alpha)V_{TOT}/N_{rest}$.. In this paper, the generated weights are further used in the gridding reconstruction program, (with a standard 3D Kaiser-Bessel window).

## Results

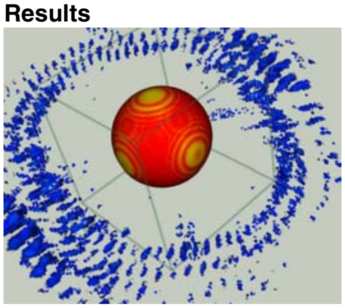

***Figure 2:** The reconstruction of a homogeneous sphere, from TPI samples. The object aliases (1% relative amplitude) define the effective field of view (gray box). Their non-trivial geometric distribution can be understood from the Voronoi cells geometry.*

Two sources of the variability in Voronoi diagram construction are extensively analyzed here, namely the maximum diameter of a cluster, $\delta$, and the volume regularization parameter $\alpha$. As an example, the twisted projection-imaging trajectory is chosen. It consists of radial lines at small K, followed by 3D curves designed such that the K-space is uniformly sampled, with a dependence of the Voronoi volume as a function of radial distance illustrated in Figure 1. The deviations from the theoretical curve make the exact computation of Voronoi factors, desirable. A different class of studies opened by this approach is the systematic study of the aliasing. As illustrated in Figure 2, where the reconstruction of a homogeneous sphere is performed from a TPI sampling, the aliases distribution is highly non-trivial, (for this particular geometry, concentrated in a plane. The substructure present in the image could be understood from the fact that the Fourier transform of polyhedral domains exhibits lines of slow decays. Whenever those lines converge, such patterns are obtained, in a manner not unlike the caustics surfaces are obtained, in optics.

## Conclusions:

We have demonstrated that by using algebraic geometry tools, the Voronoi algorithm could be implemented as a building block of a general reconstruction tool. The presented examples show the versatility of the approach, and hint to the complexity of the true 3D problems when compared to the 2D counterparts. Whenever the hardware limitations are met (limited slew rates, gradients, waveform memory) the K-space trajectory deviates from its theoretical form. The 3D Voronoi decomposition offers a natural framework to address those imprecision. Also, the nature of aliasing could be better understood in the same framework, as each individual Voronoi domain, contains information about the local structure of the grid.

## References:

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