Fast Iterative Approximate Pseudo-Inverse Image Reconstruction from Data Acquired on Arbitrary k-Space Trajectories

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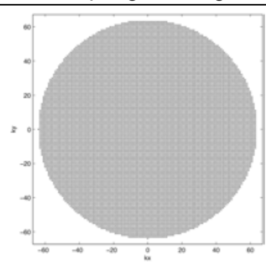
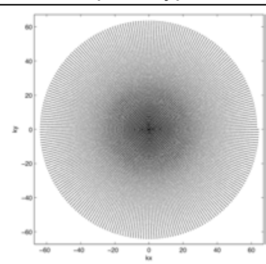
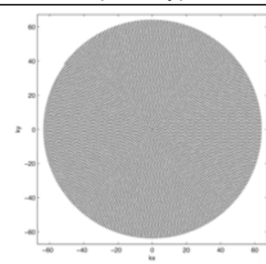
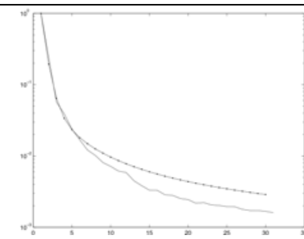
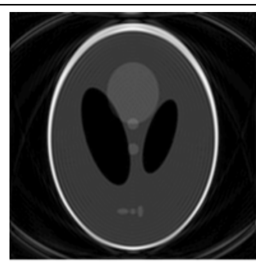
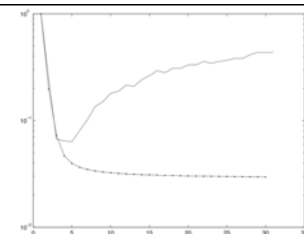
Introduction We present an efficient least squares method for the reconstruction of magnetic resonance images from data sampled non-uniformly in k-space. The procedure can be applied to arbitrary trajectories. It is based on two fast algorithms, the non-uniform fast Fourier transform (NUFFT) [1–3] and the fast sinc transform [4], combined with analytic weights [5], and an iterative procedure.

Theory and Numerical Experiments The pseudo-inverse formulation of the image reconstruction problem was presented in [6]. The forward model given by the MR signal equation is a continuous to discrete mapping from $\rho(\mathbf{x})$ to the set of N measurements $\{s(\mathbf{k})\}$. The pseudo-inverse reconstruction is given by $\rho(\mathbf{x}) = \Sigma_n e^{-2\pi i \mathbf{k_n \cdot x}} (M^+)_{nm} s_m$, where the NxN matrix M has elements $M_{mn} = \mathrm{sinc}(\mathbf{k}_n - \mathbf{k}_m)$, that is one first solves the linear system $\mathbf{s}=M\mathbf{a}$, and then one computes the fourier sum. The fourier sum can be computed quickly using the NUFFT. The linear system can be solved by a) inverting M using the SVD, an expensive procedure requiring $O(N^3)$ operations, or b) an iterative method such as conjugate gradient [7]. In [5] it was shown that the matrix M can be applied quickly using the fast sinc transform. It was also shown in [5] that an approximate inverse of M could be formed as a diagonal matrix W given by the optimal density compensation weights $w_n = \Sigma_m (1/\mathrm{sinc}^2(k_n - k_m))$, and computable quickly using the fast sinc^2 transform. Here, we combine these tools to construct a fixed point type iterative method to solve the linear system $\mathbf{s}=M\mathbf{a}$. At each iteration the signal is estimated by applying M to **a** using the fast sinc transform, and **a** is updated by multiplying the residual signal by the weights.

1. Sampling at Integers	2. Spiral Type 1	3. Spiral Type 2	4. Residual Error
			 No Noise
			 With Noise

Simulations were carried out in MATLAB and FORTRAN. Synthetic data were generated from the analytic FT of the Shepp-Logan phantom [6] on several k-space trajectories. For the samples at integers (column 1), the sinc matrix is equal to the identity and inversion is trivial. For the two archimedian spirals in columns 2 and 3 (4096 points, Kmax=64), the linear system $\mathbf{s}=M\mathbf{a}$ was solved using the either fixed point or conjugate gradient iteration. On a 1.2GHz laptop computer, each application of the sinc matrix required 0.19s. The total time for 5 iterations of sinc followed by NUFFT is ~1.5s.

<u>Gibbs Ringing</u>: A small amount of Gibbs ringing is observed in the reconstructed images due to the truncation of the sampling in the fourier domain. Kmax = 64. This is a property of any least-squares type reconstruction.

<u>Spiral Artifact</u>: The artifact in column 2 is a result of insufficient sampling at high frequency; it disappears when the spacing between the spiral turns is reduced slightly (not shown). We note that even with optimal reconstruction, some trajectories are better than others.

<u>Noise and CG</u>: If none of the sampling points are repeated, the matrix M is invertible, although it may be badly conditioned and conjugate gradient based linear solvers are potentially unstable [7]. A small amount of noise was added to the signal (same as column 3). The linear system $\mathbf{s}=M\mathbf{a}$ was solved and the log of the relative residual error as a function of iteration number was plotted in column 4 (CG: solid line, new method: line with dots). The ill-conditioned nature of the linear system becomes apparent when using conjugate gradient in the presence of even a small amount of noise (SNR~30). Our method does not suffer from this instability.

Summary We have shown that the combination of the NUFFT, the fast sinc transform, and optimal weights provides a robust iterative linear solver for the reconstruction of MR images from arbitrarily sampled k-space trajectories. This procedure is likely to be of value in a variety of fast imaging applications, such as functional or cardiac MR imaging.

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